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Re-entrant behaviour of the anisotropic BEG model in the effective-field approximation

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Abstract. The very recently improved treatment proposed by Tucker on the Honmura–Kaneyoshi exponential operator technique is herein extended to treat the anisotropic Blume–Emery–Griffiths model. It is shown that this procedure leads to an exact set of mutually coupled equations which can explicitly and systematically include effects of correlations. The method is illustrated in a honeycomb lattice by employing its simplest approximate version, in which multispin correlations are neglected. Within this framework we find that the transition temperature is double valued under certain conditions of competing bilinear and biquadratic interactions, suggesting the occurence of re-entrant behaviour in both first- and second-order phase boundary lines.

The *anisotropic* Blume–Emery–Griffiths (BEG) model (Blume *et al* 1971) is a spin-one Ising system with both bilinear and biquadratic interactions in which a single-ion uniaxial crystal field anisotropy is included. The model without single-ion anisotropy term is called the *isotropic* BEG model while the one with vanishing biquadratic interactions is often referred as the Blume–Capel (BC) model (Blume 1966, Capel 1966). Both BEG and BC models have extensively been studied in the literature (for papers published before 1984, see for instance the review article of Lawrie and Sarback 1983) because they play a fundamental role in the multicritical phenomena associated with physical systems such as ^{3}He–^{4}He mixtures, multicomponent fluids, ternary alloys, and metamagnets.

The Hamiltonian of the anistropic BEG mode is defined by

$$H = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j - \sum_{\langle i,j\rangle} J'_{ij} S_i^2 S_j^2 - D \sum_i S_i^2 \quad (J_{ij} > 0) \tag{1}$$

where J_{ij}, J'_{ij} and D are the bilinear, biquadratic and anisotropy parameters, respectively. Each S_i takes the value ± 1 and 0, and the summation is going to be carried out only over nearest-neighbour pairs of spins.

In a very recent paper Tucker (1988) has revised the application of the differential operator technique of Honmura and Kaneyoshi (1979) to the *isotropic* BEG model

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and pointed out that a previous version of the formalism presented by Siqueira and Fittipaldi (1985) should be improved if the Honmura–Kaneyoshi-type effective-field equations were treated more correctly. In fact he succeeded in taking exactly into account all the relevant local (single-site) spin kinematic relations of the spin-one subspace (such as $S_i^{2n+1} = S_i$ and $S_i^{2n} = S_i^2$; $n = 1, 2$), which has improperly been overlooked in the earlier Siqueira and Fittipaldi (1985) version of the theory. As a result he was able to generate a much more improved effective-field approximative approach which leads to results quite superior to those previously obtained within the same framework (Chakraborty 1984, 1988, Fittipaldi and Siqueira 1986, Kaneyoshi 1987). In particular, it was found by Tucker (1988) that his new treatment of the effective-field equations provides results that resemble those of the cluster variational method in pair approximation (Tucker 1987) and of other approximative procedures (De Alcantara Bonfim *et al* 1985, 1986). His work, however, was devoted only to the analysis of the *isotropic* BEG model as well as restricted to the formulation of the method up to some stage at which the possibilities of including the effects of correlations were overlooked.

The main purpose of this paper is to employ the correct procedure used by Tucker (1988) in order to present a somewhat more complete formulation of the theory in an extended spin-one Ising model described by the system Hamiltonian in (1). In contrast with the *isotropic* case, the *anisotropic* BEG model describes an interesting system since it may present, under certain conditions, a tricritical point at which the system exhibits a first-order phase transition. Here a general statistical–mechanical treatment for the anisotropic BEG model is presented, within the framework of the differential operator technique of Honmura and Kaneyoshi (1979). We point out that this method leads to an *exact* set of mutually coupled equations which is particularly amenable to systematic approximations including the effect of correlations. Here as an illustration of this scheme, the set of equations is decoupled by employing its simplest version, in which multispin correlations are neglected. By imposing the condition of vanishing anisotropy we recover the same set of effective-field equations obtained by Tucker (1988). Within this approximation we discuss the critical temperature and the tricritical point at which the system undergoes to a first-order phase transition. It should be emphasised that herein we are particularly concerned with the type of information which can be relevant in order to elucidate the predictions of the newly developed effective-field theory (Tucker 1988) as well as to investigate the effects of the single-ion anisotropy in the magnetic system under consideration. Thus, we have restricted ourselves to examine such aspects on a phase diagram of the considered *anisotropic* BEG model, on a honeycomb lattice structure.

Following the formulation of Fittipaldi and Siqueira (1986) for the *anisotropic* BEG model, the two relevant statistical–mechanical quantities $\langle S_i \rangle$ and $\langle S_i^2 \rangle$ may be evaluated from the following set of equations

$$\langle S_i \rangle = \left\langle \prod_j \exp(K_{ij} S_j \nabla_x) \prod_{j'} \exp(K'_{ij'} S_{j'}^2 \nabla_y) \right\rangle f(x, y)|_{x=0;y=0} \tag{2a}$$

$$\langle S_i^2 \rangle = \left\langle \prod_j \exp(K_{ij} S_j \nabla_x) \prod_{j'} \exp(K'_{ij'} S_{j'}^2 \nabla_y) \right\rangle g(x, y)|_{x=0;y=0} \tag{2b}$$

where $K_{ij} = \beta J_{ij}$, $K'_{ij} = \beta J'_{ij}$ (with $\beta = (k_B T)^{-1}$), $\nabla_\mu = \partial/\partial_\mu$ ($\mu = x, y$) are the two differential operators, $\langle \cdots \rangle$ denotes the standard thermal average and the functions

$f(x, y)$ and $g(x, y)$ are defined by

$$f(x,y) = \frac{2e^y \sinh x}{e^{-\beta D} + 2e^y \cosh x} \tag{3a}$$

$$g(x,y) = \frac{2e^y \cosh x}{e^{-\beta D} + 2e^y \cosh x} \tag{3b}$$

which satisfy the following conditions

$$f(x,y) = -f(-x,y) \qquad g(x,y) = g(-x,y). \tag{4}$$

By using the van der Waerden identities for the spin-one Ising spins, as shown by Siqueira and Fittipaldi (1985), equations (2) may be rewritten in a more useful form:

$$\langle S_i \rangle = \left\langle \prod_j F_i(S_j, S_j^2; \nabla_x) \prod_{j'} G_i(S_{j'}, S_{j'}^2; \nabla_y) \right\rangle f(x,y)|_{x=0;y=0} \tag{5a}$$

$$\langle S_i^2 \rangle = \left\langle \prod_j F_i(S_j, S_j^2; \nabla_x) \prod_{j'} G_i(S_{j'}, S_{j'}^2; \nabla_y) \right\rangle g(x,y)|_{x=0;y=0} \tag{5b}$$

where the operator functions $F_i(S_j, S_j^2; \nabla_x)$ and $G_i(S_j, S_j^2; \nabla_y)$ are given by

$$F_i(S_j, S_j^2; \nabla_x) = 1 + \sinh(K_{ij}\nabla_x)S_j + (\cosh(K_{ij}\nabla_x) - 1)S_j^2 \tag{6a}$$

$$G_i(S_j, S_j^2; \nabla_y) = 1 + (\exp(K'_{ij}\nabla_y) - 1)S_j^2. \tag{6b}$$

Now, as has already been successfully discussed by Tucker (1988), the two products over j and j' appearing in (5*a*) and (5*b*), which are over the same set of first-neighbour spins of site i, may be multiplied together, and by using the local spin kinematic spin-one relations referred to above, (5*a*) and (5*b*) can be recast as

$$\langle S_i \rangle = \left\langle \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \right\rangle f(x,y)|_{x=0;y=0} \tag{7a}$$

$$\langle S_i^2 \rangle = \left\langle \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \right\rangle g(x,y)|_{x=0;y=0} \tag{7b}$$

where

$$P_i(S_j, S_j^2; \nabla_x, \nabla_y) = 1 + \alpha_{ij}\gamma_{ij}S_j + (\beta_{ij}\gamma_{ij} - 1)S_j^2 \tag{8}$$

in which we have introduced the notation

$$\alpha_{ij} = \sinh(K_{ij}\nabla_x) \qquad \beta_{ij} = \cosh(K_{ij}\nabla_x) \qquad \gamma_{ij} = \exp(K'_{ij}\nabla_y). \tag{9}$$

We should note that (7*a*) and (7*b*) with $D = 0$ reduce to the equations obtained by Tucker (1988) if correlations between different spins are neglected at this stage of the formalism.

At this point it is worth mentioning that the *exact* set of equations (7*a*) and (7*b*) is particularly amenable to the use of various approximate schemes which can explicitly and systematically include effects of correlations. Firstly, we should note that (7*a*) and (7*b*) provide a set of relations between the quantities $\langle S_i^n \rangle$ $(n = 1, 2)$ and associated multispin correlation functions of the various sites. In order to illustrate such a type of exact relations, we now particularise (7*a*) and (7*b*) for the case of a honeycomb lattice structure, in which by performing a tedious but straightforward calculation yields the following expressions for $\langle S_i \rangle$ and $\langle S_i^2 \rangle$:

$$\langle S_i \rangle = \Bigg[\sum_{j=1}^{3} \Bigg(\gamma_{ij}\alpha_{ij}\langle S_j \rangle + \sum_{k(\neq j)} \gamma_{ij}\alpha_{ij}(\gamma_{ik}\beta_{ik} - 1)\langle S_j S_k^2 \rangle + \frac{1}{2!}\sum_{k(\neq j)}\sum_{\ell(\neq j,k)} \gamma_{ij}\alpha_{ij}(\gamma_{ik}\beta_{ik} - 1)(\gamma_{i\ell}\beta_{i\ell} - 1)\langle S_j S_k^2 S_\ell^2 \rangle + \frac{1}{3!}\sum_{k(\neq j)}\sum_{l(\neq j,k)} \gamma_{ij}\alpha_{ij}\gamma_{ik}\alpha_{ik}\gamma_{il}\alpha_{il}\langle S_j S_k S_l \rangle \Bigg) \Bigg] f(x, y)|_{x=0;y=0} \qquad (10a)$$

$$\langle S_i^2 \rangle = \Bigg[1 + \sum_{j=1}^{3} \Bigg((\gamma_{ij}\beta_{ij} - 1)\langle S_j^2 \rangle + \frac{1}{2!}\sum_{k(\neq j)} \gamma_{ij}\alpha_{ij}\gamma_{ik}\alpha_{ik}\langle S_j S_k \rangle + \frac{1}{2!}\sum_{k(\neq j)} (\gamma_{ij}\beta_{ij} - 1)(\gamma_{ik}\beta_{ik} - 1)\langle S_j^2 S_k^2 \rangle + \frac{1}{2!}\sum_{k(\neq j)}\sum_{\ell(\neq j,k)} \gamma_{ij}\alpha_{ij}\gamma_{ik}\alpha_{ik}(\gamma_{i\ell}\beta_{i\ell} - 1)\langle S_j S_k S_\ell^2 \rangle + \frac{1}{3!}\sum_{k(\neq j)}\sum_{l(\neq j,k)} (\gamma_{ij}\beta_{ij} - 1)(\gamma_{ik}\beta_{ik} - 1)(\gamma_{il}\beta_{il} - 1)\langle S_j^2 S_l^2 S_k^2 \rangle \Bigg) \Bigg] g(x, y)|_{x=0;y=0}. \qquad (10b)$$

In arriving at this set of equations, we have used properties of the exponential operators, such as $\Phi_{\text{even}}(\nabla_x) f(x, y)|_{x=0;y=0} = 0$ and $\Phi_{\text{odd}}(\nabla_x) g(x, y)|_{x=0;y=0} = 0$, valid for any *even* and *odd* functional $\Phi(\nabla_x)$, which are derived by using relations (4). In earlier works (Kaneyoshi *et al* 1981, Taggart and Fittipaldi 1982) on the two-state Ising models, the effects of correlations has successfully been included by the use of an analogous set of exact identities. For this purpose, one should generate a new set of exact formal identities for all the relevant multispin correlations appearing in the right-hand sides of both (10*a*) and (10*b*), by using the following generalised spin identities

$$\langle \{i\} S_i \rangle = \Big\langle \{i\} \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \Big\rangle f(x, y)|_{x=0;y=0} \qquad (11a)$$

$$\langle \{i\} S_i^2 \rangle = \Big\langle \{i\} \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \Big\rangle g(x, y)|_{x=0;y=0} \qquad (11b)$$

in which $\{i\}$ denotes any function of Ising variables at sites other than the *i*th. Thus, in a similar way, equations (10*a*) and (10*b*) can be used as a basis for various approximation schemes which may explicitly take into account correlation effects. Here, as an illustration of the method, we restrict ourselves to the simplest approximation in which all high-order spin correlations on the right-hand sides of (10*a*) and (10*b*) are neglected. It is clear that within this approximation the strict criticality of the system is lost (in particular, the critical exponents are going to be the classical ones), and the real dimensionality of the system is only partially incorporated through the coordination number of the lattice. Nevertheless, as has already been discussed in several works on spin-one Ising systems (Tucker 1988, Siqueira and Fittipaldi 1986, Kaneyoshi 1986), such a framework is quite superior to the ordinary mean field approximation (MFA) and provides in particular a vanishing critical temperature for one-dimensional systems. This is so because in this type of treatment relations such as $S_i^2 = 0, 1$ as well as $S_i^3 = S_i$ and $S_i^4 = S_i^2$, are taken exactly into account, while in the usual MFA all the self- and multi-spin correlations are neglected.

Based on this approximation (i.e. decoupling the multispin correlations as $\langle S_j S_k S_\ell^2\rangle \simeq \langle S_j\rangle\langle S_k\rangle\langle S_\ell^2\rangle; \langle S_j S_k^2 S_\ell^2\rangle \simeq \langle S_j\rangle\langle S_k^2\rangle\langle S_\ell^2\rangle$ and so on) the magnetisation $m = \langle S_i\rangle$ and the quadrupolar moment $q = \langle S_i^2\rangle$ can be evaluated from the following set of mutually coupled equations

$$m = 3A_{13}m + A_{33}m^3 \tag{12a}$$

$$q = B_{03} + 3B_{23}m^2 \tag{12b}$$

where the coefficients $A_{vz}(q, T)$ and $B_{vz}(q, T)$, which are q- and T-dependent (with the exception of A_{33}), are given by

$$A_{13} = \exp(K'\nabla_y)\sinh(K\nabla_x)[1 + q(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]^2 f(x, y)|_{x=0;y=0} \tag{13a}$$

$$A_{33} = \exp(3K'\nabla_y)\sinh^3(K\nabla_x) f(x, y)|_{x=0;y=0} \tag{13b}$$

and

$$B_{03} = [1 + q(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]^3 g(x, y)|_{x=0;y=0} \tag{14a}$$

$$B_{23} = \exp(2K'\nabla_y)\sinh^2(K\nabla_x)[1 + q(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]g(x, y)|_{x=0;y=0}. \tag{14b}$$

Here the subscripts v and z denote the power of m and the coordination number, respectively. The set of equations (12) can easily be generalised for arbitrary coordination number z.

Now let us address our attention to the study of the transition temperature and the tricritical point of the system. By expanding in an interactive procedure the RHS of (12*a*), by using (12*b*) one obtains in general an equation for m of the form

$$m = am + bm^2 + cm^3 + \cdots. \tag{15}$$

The second-order phase transition line is then determined by $1 = a$, i.e.

$$1 = 3\exp(K'\nabla_y)\sinh(K\nabla_x)[1 + q_o(\exp(K'\nabla_y)\cosh(K\nabla_y) - 1)]^2 f(x, y)|_{x=0;y=0} \tag{16}$$

where q_o is the solution of

$$q_o = [1 + q_o(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]^3 g(x, y)|_{x=0;y=0}. \tag{17}$$

In the vicinity of the second-order phase transition line, the magnetisation m is given by

$$m^2 = (1 - a)/b. \tag{18}$$

The RHS must be positive. If this is not the case, the transition is of the first order, and hence the point at which $a = 1$ and $b = 0$ is the tricritical point (Benayard *et al* 1985).

At this point, in order to obtain the expression for b, let us substitute

$$q = q_o + q_1 m^2 \tag{19}$$

into (12b). The expression for q_1 is then given by

$$q_1 = f/(1 - e) \tag{20}$$

with

$$f = 3\exp(2K'\nabla_y)\sinh^2(K\nabla_x)[1 + q_o(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]g(x, y)|_{x=0;y=0} \tag{21}$$

$$e = 3[\exp(K'\nabla_y)\cosh(K\nabla_x) - 1][1 + q_o(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]^2 g(x, y)|_{x=0;y=0}. \tag{22}$$

Substituting (19) into (12a), the expression of b in (15) is given by

$$\begin{aligned} b = 6q_1 \exp(K'\nabla_y)\sinh(K\nabla_x)(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1) \\ \times [1 + q_o(\exp(K'\nabla_y)\cosh(K\nabla_x) - 1)]f(x, y)|_{x=0;y=0} \\ + \exp(3K'\nabla_y)\sinh^3(K\nabla_x)f(x, y)|_{x=0;y=0}. \end{aligned} \tag{23}$$

Thus equations (16), (17), (20), and (23) are the expressions of the anisotropic BEG model with $z = 3$ for evaluating the second-order phase transition line and the tricritical point. The equations can be easily calculated by the use of a mathematical relation $\exp(\lambda\nabla_\mu)\phi(\mu) = \phi(\mu + \lambda)$.

We have solved numerically the relation ($a = 1$, with $b < 0$) and the critical condition ($a = 1$ and $b = 0$, with $c < 0$), which yield, respectively, the critical frontiers which separate the ferromagnetic phase from any other phase and the tricritical point at which the phase transition changes from second to first order. Moreover, in order to determine the first-order transition lines one can, for instance, proceed in the following way. Firstly, one considers the BEG model in the presence of an external magnetic field, H_0, by adding a term $-\mu H_0 \sum_i S_i$ to the system Hamiltonian in (1), where μ is the Landé factor times the Bohr magneton. The magnetisation $m = \langle S_i \rangle$ and the quadrupolar moment $q = \langle S_i^2 \rangle$ are now given by the same set of relations (7) in which the variable x, appearing in function $f(x, y)$ and $g(x, y)$ defined in (3), is replaced by $x + h$, where $h = \mu\beta H_0$. With this modification the symmetry properties associated with the variable x pointed out in (4) are then broken. Accordingly, in order to evaluate thermodynamical properties it is useful to expand the RHS of (7a) and (7b) with respect to h and retain only its first-order terms. If this is done, then one may write

$$\langle S_i \rangle = \langle \hat{F} \rangle + \langle \hat{K} \rangle h \tag{24a}$$

$$\langle S_i^2 \rangle = \langle \hat{G} \rangle + \langle \hat{L} \rangle h \tag{24b}$$

where

$$\hat{F} = \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) f(x, y)|_{x=0;y=0}$$
$$\hat{G} = \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) g(x, y)|_{x=0;y=0} \tag{25a}$$

$$\hat{K} = \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \tilde{g}(x, y)|_{x=0;y=0}$$
$$\hat{L} = \prod_j P_i(S_j, S_j^2; \nabla_x, \nabla_y) \tilde{f}(x, y)|_{x=0;y=0} \tag{25b}$$

in which $f(x, y)$ and $g(x, y)$ are defined as above (equations (3*a*) and (3b)), and

$$\tilde{f}(x, y) = f(x, y)[1 - g(x, y)] \qquad \tilde{g}(x, y) = g(x, y)[1 - f(x, y) \tanh x]. \tag{26}$$

Hence, from equations (24*a*) and (24b) it is seen that the presence of the magnetic field introduces to the RHS of equations (10*a*) and (10*b*) *even* and *odd* correlation functions respectively. Then the formalism is developed in the same way as before and it is found that, by the use of the same sort of approximate scheme (in which multispin correlations are neglected), one derives equations analogous to (12). With this new set of equations (in which the coefficients of the even and odd powers of m on the RHS of (12*a*) and (12*b*) respectively are no longer null), one obtains the equation of state $m(T, H)$. In the case of a first-order phase transition, the isotherms in the $(m-H)$ plane have the same typical shape as that of the van der Waals isotherms (see, for instance, Stanley 1971) and, as usual, the position of these first-order transitions is obtained by applying the 'equal areas' Maxwell construction, from which T_c as a function of D/J (for fixed values of J'/J) is located in zero field. Here, for brevity, is not presented great details of such a procedure, referring the reader instead to the book by Reichl (1987).

The phase diagram in the T–D plane is presented in figures 1 and 2, for corresponding typical fixed *negative* values of the ratio $J'/J = \alpha$. As a first observation, we should note that the nature of variations of T_c versus D reveal a common basic behaviour, which is the fact that all the critical transition lines decrease when D/J decreases, reaching the zero temperature limit at distinct values of D/J. These results revise our predictions in figure 1 of the recent paper of Kaneyoshi *et al* (1988).

As one can see from figure 1, the critical frontiers in the T–D plane present a peculiar behaviour. We find that for $-1.0 < \alpha < 0$ the system exhibits tricritical points (full circles) and the first-order segments (broken regions of the curves with $\alpha = -0.5$ and $\alpha = -0.75$) of the ferromagnetic phase boundary lines shows bulges, suggesting the occurence of the re-entrant phenomena. The situation, however, abruptly changes at $\alpha = -1.0$. For $\alpha < -1.0$ the tricritical point does not appear, but a re-entrant behaviour, now associated with the second-order transition lines, occurs as shown in the enlarged-scale curves of figure 2. The region of the phase diagram in the T–D plane in which competing bilinear and biquadratic interactions may occur (i.e., $\alpha < 0$) and $-1.0 < D/J < 1.0$ is, indeed, quite interesting and has been missed in most of the works published so far. Furthermore, the fact that the second-order phase boundary line labelled with $\alpha = -1.0$ in figure 1 reaches the zero-temperature limit at

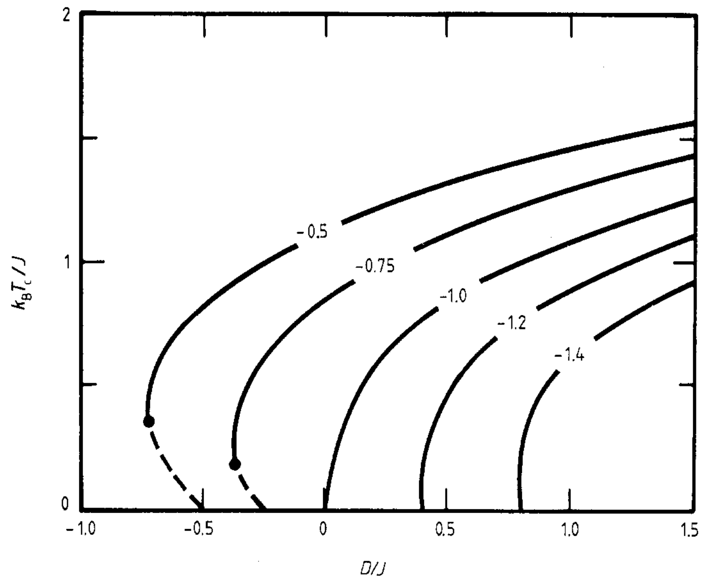

Figure 1. Nature of variation of the critical temperature (k_BT_c/J) with respect to the reduced anisotropy parameter D/J for the honeycomb lattice. Numerical figures associated with each curve are the various values of the reduced biquadratic parameter $\alpha = J'/J$. The full circles denote the tricritical points. The continuous and broken regions of the curves refer to the second- and first-order phase transitions, respectively.

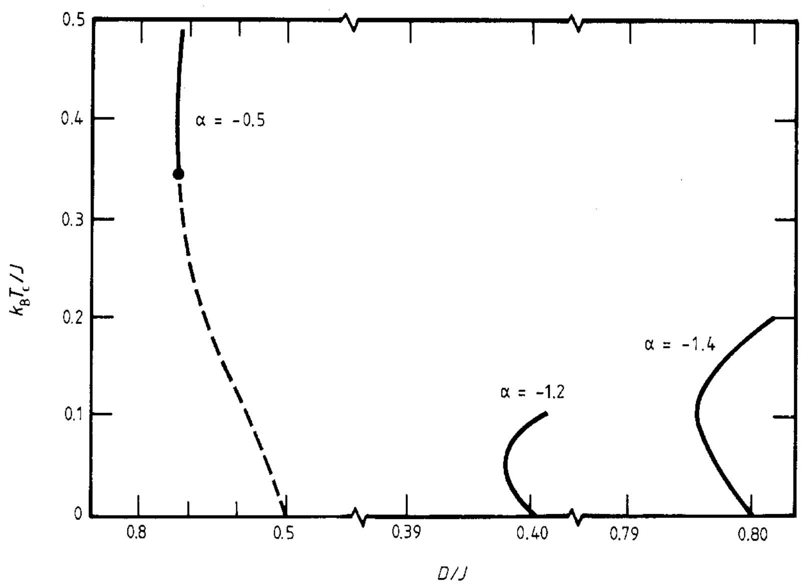

Figure 2. As in figure 1, but with enlarged scales in order to emphasise the re-entrant behaviour of curves labelled in figure 1 with $\alpha = -1.4$, $\alpha = -1.2$ and $\alpha = -0.5$.

$D = 0$, in complete agreement with Monte Carlo simulations (De Alcantara Bonfim and Obcemea 1986, Wang *et al* 1987), mean field renormalisation group technique (De Alcantara Bonfim and Sá Barreto 1985) as well as with the *exact* solution of the model on the Bethe lattice (Chakraborty and Tucker 1986), gives to the present treatment some qualitative and, to a certain extent, quantitative confidence.

In conclusion, the applicability of the correct treatment proposed by Tucker (1988) on the Honmura–Kaneyoshi exponential operator technique for the *anisotropic* BEG model has been demonstrated and used to investigate, in its simplest approximate version, the nature of variation of the critical temperature in the T–D plane as well as the dependence of the position of the tricritical point, on a honeycomb lattice. It is shown that this effective-field treatment (Tucker 1988) leads to the conclusion that the transition temperature has bulges for negative α (except for $\alpha = -1.0$), suggesting the occurence of re-entrant phenomena.

Finally, we would like to conclude by briefly mentioning that after this work was finished we received a preprint of a Comment by Dr J W Tucker in which he also addressed himself to the study of the tricritical behaviour of the *anisotropic* BEG model by means of the recently developed effective-field theory (Tucker 1988) for the honeycomb, square and cubic lattices. His work, however, is restricted to the presentation of numerical results for the dependence of the position of the tricritical point for a range of biquadratic exchange strengths where the critical frontier for the second-order transition is relatively well behaved. Here, besides the analysis of the dependence of the tricritical point we have also investigated the phase diagram in the T–D plane for both first- and second-order transition lines. We believe that not only would the present study put to test the proposed (Tucker 1988) effective-field theory, but would also help to provide an independent contribution to the present problem.

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